Math 246B Lecture 11 Notes

Daniel Raban

February 1, 2019

1 Runge's Theorem and Compact Exhaustion

1.1 Runge's theorem

Last time, we showed that if $\Omega \subseteq \mathbb{C}$ is open, $K \subseteq \Omega$ is compact, and $f \in \mathrm{Hol}(\Omega)$, then

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \psi}{\partial \bar{\zeta}} \frac{f(\zeta)}{\zeta - z} \, L(ds),$$

where $\psi \in C_0^1(\Omega)$ and $\psi = 1$ near K.

Let $\Omega \subseteq \mathbb{C}$ be open, and let $\tilde{\Omega} \subseteq \Omega$ be a connected component of Ω. Then $\tilde{\Omega}$ is open, and $\partial\tilde{\Omega} \subseteq \partial\Omega \subseteq \mathbb{C} \setminus \Omega$.

Example 1.1. Let $K \subseteq \mathbb{C}$ be compact, and let $\Omega = \mathbb{C} \setminus K$. Then Ω has precisely 1 unbounded component. Indeed, if $R > 0$ is large, then $\{|z| > R\} \subseteq \Omega$ is connected, so it is contained in a single component.

Theorem 1.1 (Runge)**.** *Let $K \subseteq \mathbb{C}$ be compact, and let $A \subseteq \mathbb{C}$ be such that any bounded component of $\mathbb{C} \setminus K$ intersects A. Let f be holomorphic in a neighborhood of K. Then for every $\varepsilon > 0$, there is a rational function $r(z) = p(z)/q(z)$ with p, q polynomials and $q(z) \neq 0$ (when $z \notin A$) such that $|f(z) - r(z)| \leq \varepsilon$ for all $z \in K$.*

Proof. We can use the previous formula for f, where Ω is our neighborhood of K where f is holomorphic. Approximate the right hand side by a Riemann sum of the form

$$g(z) = \sum_j \frac{a_j}{\zeta_j - z},$$

where $\zeta_j \in \mathbb{C} \setminus K$. Then approximate each $1/(\zeta_j - z)$ by a rational function as in the theorem, using a "pole-pushing" argument. By approximating with suitable polynomials, we can "push" the pole from ζ_j to another point outside of A. □

Corollary 1.1 (Runge's theorem for polynomials)**.** *Let $K \subseteq \mathbb{C}$ be compact and simply connected, and let f be holomoprhic in a neighborhood of K. Then f can be approximated by polynomials in z, uniformly on K.*

Remark 1.1. The condition that A meets every bounded component of $\mathbb{C}\setminus K$ is necessary. Let V be a bounded component of $\mathbb{C}\setminus K$, ket $a \in V$, and let $f(z) = \frac{1}{z-a}$ be holomorphic in a neighborhood of K. Assume that for every $\varepsilon > 0$, there exists r(z) rational with no poles in V such that $|f(z) - r(z)| \leq \varepsilon$ on K. Then $|1-(z-a)r(z)| \leq C\varepsilon$ for all $z \in K$. Now $\partial V \subseteq K$, so, by the maximum principle, $|1-(z-a)r(z)| \leq C\varepsilon$ for all $z \in V$. This is a contradiction when we set $z = a$.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open, and let $\omega \subseteq \Omega$ be open. Then ω is **relatively compact** if $\overline{\omega}$ is a compact subset of Ω.

Corollary 1.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $K \subseteq \Omega$ be compact. Assume that no component of $\Omega \setminus K$ is relatively compact in Ω. Then any function holomorphic in a neighborhood of K can be approximated uniformly on K by functions in* $\mathrm{Hol}(\Omega)$.

Proof. In view of Runge's theorem, we only need to check that if O is a bounded component of $\mathbb{C}\setminus K$, then $O \cap (\mathbb{C}\setminus\Omega) \neq \varnothing$. Indeed, if $O \subseteq \Omega$, then $\overline{O} \subseteq \Omega$. Here, $\overline{O}$ is compact, and O is a component of $\Omega \setminus K$. □

1.2 Compact exhaustion

Proposition 1.1 (compact exhaustion with good properties)**.** *Let $\Omega \subseteq \mathbb{C}$ be open. There exist compact sets $K_n \subseteq \Omega$ such that*

1. *$K_n \subseteq K_{n+1}$ for $n = 1, 2, \dots$.*
2. *$\bigcup_{n=1}^{\infty} K_n = \Omega$.*
3. *Every bounded component of $\mathbb{C}\setminus K_n$ intersects $\mathbb{C}\setminus\Omega$.*

Proof. Set $K_n = \{z \in \mathbb{C} : |z| \leq n, \mathrm{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/n\}$. Then we have the first two properties. Let us check that each bounded component of $\mathbb{C}\setminus K_n$ contains a bounded component of $\mathbb{C}\setminus\Omega$.

$$\begin{aligned} \mathbb{C}\setminus K_n &= \{|z| > n\} \cup \{z : \mathrm{dist}(z, \mathbb{C}\setminus\Omega) < 1/n\} \\ &= \{|z| > n\} \cup \bigcup_{a \in \mathbb{C}\setminus\Omega} D(a, 1.n). \end{aligned}$$

Let O be a bounded component of $\mathbb{C}\setminus K_n$. Then $O \subseteq \bigcup_{a\in\mathbb{C}\setminus\Omega} D(a, 1/n)$. Thus, there exists $a \in \mathbb{C}\setminus\Omega$ such that $D(a, 1/n) \subseteq \Omega$. Let V be the component of $\mathbb{C}\setminus\Omega$ such that $a \in V$. Then $V \subseteq \mathbb{C}\setminus\Omega \subseteq \mathbb{C}\setminus K_n$ is connected, and $V \cap O \neq \varnothing$. Thus, $V \subseteq O$, so V is bounded. □

Next time, we will show that if $f \in \mathrm{Hol}(\Omega)$, there exist rational r_n, holomoprhic in Ω, such that $r_n \to f$ locally uniformly.